國立高雄大學九十九學年度研究所碩士班招生考試試題

科目：基礎數學　　系所組別：統計學研究所統計組　　是否使用計算機：否
考試時間：100 分鐘　　本科原始成績：100 分

1. (15%) Let X be an $n \times k$ matrix with $\text{rank}(X) = k$. Define $P = X(X^TX)^{-1}X^T$, where X^T denotes the transpose of X. Answer the following questions:
   (a) (5%) Is P an idempotent matrix?
   (b) (5%) Find the rank of P.
   (c) (5%) Find all possible eigenvalues of P.

2. (15%) Let
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$
   (a) (5%) Show that A is positive definite.
   (b) (10%) Find a matrix $A^{1/2}$ such that $A^{1/2}A^{1/2} = A$.

3. (10%) Let A be a symmetric matrix of order $n \times n$. Show that
$$\lambda_{(1)} \le \frac{x^TAx}{x^Tx} \le \lambda_{(n)}$$
   for any nonzero vector x, where $\lambda_{(1)}$ and $\lambda_{(n)}$ denote the smallest and largest eigenvalues of A, respectively.

4. (10%) Let $T : R_n \to R_m$ be represented by an $n \times m$ matrix of rank ρ.
   (a) (5%) Show that $\dim[T(R_n)] = \rho$.
   (b) (5%) Show that if $n \le m$ and $\rho = n$, then T is one-to-one.

5. (10%) Find the area of the largest rectangle that can be inscribed in a semicircle of radius 2.

6. (12%) Find the 4th degree Taylor polynomials centered at 0 for the functions $f_1(x) = \cos x$, $f_2(x) = \ln(1 + x)$ and $f_3(x) = \sinh x$, respectively.

7. (28%) Evaluate the following:
   (a) (7%) $\lim_{x\to 5} \frac{x^2}{x-5} \int_5^x \frac{\tan u - \sin u}{u} du$
   (b) (7%) $\lim_{x\to\infty} \left(1 + \frac{2}{x+1}\right)^{\ln x}$
   (c) (7%) $\int_0^{\sqrt{3}} \frac{x}{x^4+9} dx$
   (d) (7%) $\int_0^1 \ln x dx$