國立高雄大學九十八學年度碩士班招生考試試題

科目：基礎數學
考試時間：100 分鐘

系所：
統計學研究所統計組
本科原始成績：100 分

是否使用計算機：否

1. (10 points) Compute $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$.

2. (14 points)

   (a) Show that
   $$\int_0^\pi x f(\sin x)\ dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\ dx.$$

   (b) Use part (a) to deduce the formula
   $$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\ dx = \pi\int_0^1 \frac{dx}{1+x^2}.$$

3. (10 points) Use integration by parts to derive the recursion formula
$$\int \cos^n x\ dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\ dx.$$

4. (14 points) Given that $\sum_{n=0}^\infty x^n/n! = e^x$ for all x, find the sum of the following series, assuming it is permissible to operate on infinite series as though they were finite sum.

   (a) $\sum_{n=2}^\infty \frac{n-1}{n!}$;

   (b) $\sum_{n=2}^\infty \frac{(n-1)(n+1)}{n!}$.

5. (10 points) Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Prove that $A^2 = 2A - I_2$ and compute A^{100}, where I_2 is a 2×2 identity matrix.

6. (18 points) Determine the inverse of each of the following matrices.
$$\begin{pmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

7. (12 points) Verify the following matrix is positive definite.
$$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

8. (12 points) Let A be an $n\times n$ symmetric matrix such that $A^t = A^{t+1}$ for some positive integer t, $t > 2$. Show that A is an idempotent matrix, i.e. $A^2 = A$.