國立高雄大學九十八學年度博士班招生考試試題

科目：機率論
考試時間：100 分鐘

系所：
統計學研究所
本科原始成績：100 分

是否使用計算機：否

1. Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be sequences of random variables and X and Y be random variables. Please indicate whether the following statements are true or false. If a statement is false, explain why. (30%)
   (a) If $X_n \to X$ in probability and $Y_n \to Y$ in probability, then $X_n + Y_n \to X + Y$ in probability.
   (b) If $X_n \to X$ in distribution and $Y_n \to Y$ in distribution, then $X_n + Y_n \to X + Y$ in distribution.
   (c) If $E|X_n - X| \to 0$ as $n \to \infty$, then $X_n \to X$ in probability.
   (d) If $X_n \to X$ in probability, then $E|X_n - X| \to 0$ as $n \to \infty$.
   (e) If $X_n \to X$ almost surely, then $X_n \to X$ in probability.
   (f) If $X_n \to X$ in probability, then $X_n \to X$ almost surely.
2. A sequence of random variables $\{X_n, n \geq 1\}$ is called uniformly integrable if for every $\varepsilon > 0$, there corresponds a $\delta > 0$ such that

$$\sup_{n \geq 1} \int_A |X_n| \, dP < \varepsilon,$$

whenever $P(A) < \delta$, and, in addition,

$$\sup_{n \geq 1} E|X_n| < \infty.$$

Show that if $\{X_n, n \geq 1\}$ is uniformly integrable, then $\{S_n/n, n \geq 1\}$ is also uniformly integrable, where $S_n = \sum_{i=1}^{n} X_i$. (20%)

3. Let X and Y be random variables and $\{X_n, n \geq 1\}$ be a sequence of random variables. Define

$$\rho(X) = E\left(\frac{|X|}{1+|X|}\right).$$

Show that
   (a) $\rho(X+Y) \leq \rho(X) + \rho(Y)$, (5%)
   (b) $\rho(\sigma X) \leq \max\{|\sigma|, 1\}\rho(X)$, where σ is a real number, (5%)
   (c) $\lim_{n\to\infty} \rho(X_n - X) = 0$ if and only if $X_n \to X$ in probability. (15%)
4. Let $X_1, X_2, \cdots$ be independent identically distributed random variables with mean 0, variance 1, and $EX_i^4 < \infty$. Find the limiting distribution of

$$Z_n = \sqrt{n}\,\frac{X_1X_2 + X_3X_4 + \cdots + X_{2n-1}X_{2n}}{X_1^2 + X_2^2 + \cdots + X_{2n}^2}. \qquad (25\%)$$