國立高雄大學一百學年度研究所碩士班招生考試試題

科目：數理統計
考試時間：100 分鐘

系所：
統計學研究所(統計組)
本科原始成績：100 分

是否使用計算機：否

1. (15%) Let (X,Y) be random variables with joint density

$$f(x,y) = 360xy^2(1-x-y),\ x>0,\ y>0,\ x+y\le 1.$$

   (a) Find the conditional density function of Y given $X=x$, $f(y|x)$.

   (b) Compute $\text{Var}(Y|x)$.

2. (15%) Let $X_1,\cdots,X_n$ be a random sample from the density

$$f(x) = \lambda e^{-\lambda x},\ x>0,\ \lambda>0.$$

   Let $Z_n = X_{(1)}$ be the smallest order statistics.

   (a) Find the limiting distribution of Z_n.

   (b) Find the limiting distribution of nZ_n.

3. (15%) Let $X_1,\cdots,X_n$ be a random sample from the uniform distribution on the interval $(\theta,\theta+1)$, $-\infty<\theta<\infty$. Find a minimal sufficient statistic for θ. Is it complete? Why or why not?

4. (15%) Let $X_1,\cdots,X_n$ be a random sample from $N(\mu,\sigma^2)$ distribution, $\mu\in R$, $\sigma^2>0$, where μ and σ^2 are both unknown. Find the UMVUE (uniformly minimum variance unbiased estimator) of σ^2. Does its variance attain the CRLB (Cramér-Rao lower bound)?

5. (20%) Let $X_1,\cdots,X_n$ be a random sample from the Poisson distribution with intensity rate $\lambda>0$. Find the MLE (maximum likelihood estimator) and the UMVUE of $P(X_1=0)$.

6. (20%) Let $X_1,\cdots,X_n$ be a random sample from $N(\mu,\sigma^2)$ distribution, where σ^2 is unknown. Consider testing $H_0:\mu=\mu_0$ vs. $H_1:\mu\ne\mu_0$.

   (a) Does a UMP (uniformly most powerful) test exist? Why or why not?

   (b) Show that the test that rejects H_0 when

$$|\overline{X}-\mu_0| > t_{n-1,\alpha/2}\sqrt{S^2/n}$$

   can be derived as an LRT (likelihood ratio test), where $t_{n-1,\alpha/2}$ satisfies $P(T_{n-1}\ge t_{n-1,\alpha/2})=\alpha/2$ with T_{n-1} following the t distribution with $n-1$ degrees of freedom, $\overline{X}$ is the sample mean and S^2 is the sample variance.