國立高雄大學九十九學年度研究所碩士班招生考試試題

科目：機率論　　系所組別：統計學研究所統計組　　是否使用計算機：否
考試時間：100 分鐘　　本科原始成績：100 分

1. (12 points) For any two random variables X and Y with finite variances, prove that (a) $\text{Cov}(X, Y) = \text{Cov}(X, E(Y|X))$ and (b) X and $Y - E(Y|X)$ are uncorrelated.

2. (20 points) Let $(X_1, P_1), \ldots, (X_k, P_k)$ be independent random variables with

$$X_i|P_i \sim \text{binomial}(n_i, P_i), i = 1, \ldots, k,$$

$$P_i \sim \text{beta}(\alpha, \beta).$$

Define $Y = \sum_{i=1}^k X_i$. Show that $E(Y) = \frac{\alpha}{\alpha+\beta}\sum_{i=1}^k n_i$ and $Var(Y) = \sum_{i=1}^k Var(X_i)$, where

$$Var(X_i) = n_i \frac{\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

3. (12 points) Let (X_1, X_2, X_3, X_4) have joint pdf

$$f_{\mathbf{X}}(x_1, x_2, x_3, x_4) = 24\exp(-x_1 - x_2 - x_3 - x_4), 0 < x_1 < x_2 < x_3 < x_4 < \infty.$$

Consider the transformation

$$U_1 = X_1, U_2 = X_2 - X_1, U_3 = X_3 - X_2, U_4 = X_4 - X_3.$$

Find the marginal distributions of $U_i, i = 1, \ldots, 4$ and show that U_1, U_2, U_3, U_4 are mutually independent.

4. (12 points) Let $X_1, \ldots, X_n$ be iid uniform(0,1). Let $X_{(j)}, j = 1, 2, \ldots, n$, denote the order statistics of $X_1, \ldots, X_n$. Show that $E(X_{(j)}) = \frac{j}{n+1}$ and $Var(X_{(j)}) = \frac{j(n-j+1)}{(n+1)^2(n+2)}$.

5. (18 points) Let $X_i, i = 1, 2, 3$, be independent with normal(i, i^2) distributions. For each of the following situations, use X_i's to construct a statistic with the indicated distribution.
   (a) chi squared with 3 degrees of freedom
   (b) t distribution with 2 degree of freedom
   (c) F distribution with 1 and 2 degrees of freedom

6. (12 points) Let X be a random variable with an $F_{p,q}$ distribution. Show that $(p/q)X/[1 + (p/q)X]$ has a beta distribution with parameters $p/2$ and $q/2$.

7. (14 points) Let $X_1, X_2, \ldots$ be a sequence of iid random variables whose moment generating functions, $M_{X_i}(t)$, exist in a neighborhood of 0. Let $E(X_i) = \mu$ and $Var(X_i) = \sigma^2 > 0$. Define $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$. Show that $\sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limiting standard normal distribution.