國立高雄大學九十八學年度碩士班招生考試試題

科目：微積分
考試時間：100 分鐘

系所：
統計學研究所風險管理組
本科原始成績：100 分

是否使用計算機：否

1. (14 points) Compute each of the following integrals.

   (a) $\int_0^2 f(x)dx$ where $f(x) = \begin{cases} x^2, & \text{if } 0 \leq x \leq 1; \\ 2-x, & \text{if } 1 \leq x \leq 2. \end{cases}$

   (b) $\int_0^1 f(x)dx$ where $\begin{cases} x, & \text{if } 0 \leq x \leq c; \\ c\frac{1-x}{1-c}, & \text{if } c \leq x \leq 1, \end{cases}$ c is a fixed real number and $0 < c < 1$.

2. (10 points) Compute $\lim_{x\to 0} \frac{1-\sqrt{1-x^2}}{x^2}$.

3. (14 points) Prove the following inequalities:

$$\frac{11}{24} \leq \int_0^{1/2} \sqrt{1-x^2}\, dx \leq \frac{11}{24}\sqrt{\frac{4}{3}}.$$

4. (14 points)

   (a) Show that

$$\int_0^{\pi} xf(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx.$$

   (b) Use part (a) to deduce the formula

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \pi \int_0^1 \frac{dx}{1+x^2}.$$

5. (10 points) Use integration by parts to derive the recursion formula

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx.$$

6. (14 points) For what value of a will $x^{-2}(e^{ax} - e^x - x)$ tend to a finite limit as $x \to 0$? What is the value of this limit?

7. (10 points) Assume $\lim_{n\to 0} a_n = 0$. Use the definition of limit to prove that $\lim_{n\to 0} a_n^2 = 0$.

8. (14 points) Given that $\sum_{n=0}^{\infty} x^n/n! = e^x$ for all x, find the sum of the following series, assuming it is permissible to operate on infinite series as though they were finite sum.

   (a) $\sum_{n=2}^{\infty} \frac{n-1}{n!}$;

   (b) $\sum_{n=2}^{\infty} \frac{(n-1)(n+1)}{n!}$.