科目：統計學　　系所組別：統計學研究所風險管理組　　是否使用計算機：否
考試時間：100 分鐘　　本科原始成績：100 分

1. Of the travelers arriving at a small airport, 60% fly on major airlines, 30% fly on privately owned planes, and the remainder fly on commercially owned planes not belonging to a major airline. Of those traveling on major airlines, 50% are traveling for business reasons, whereas 60% of those arriving on private planes and 90% of those arriving on other commercially owned planes are traveling for business reasons. Suppose that we randomly select one person arriving at this airport. What is the probability that the person
   (a) is traveling on business?
   (b) is traveling for business on a privately owned plane?
   (c) arrived on a privately owned plane, given that the person is traveling for business reasons?
   (d) is traveling on business, given that the person is flying on a commercially owned plane? (12%)

2. Let Y be a random variable such that

$$P(-1)=\frac{1}{18},\qquad P(0)=\frac{16}{18},\qquad \text{and}\qquad P(1)=\frac{1}{18}.$$

   (a) Show that $E(Y)=0$ and $V(Y)=\frac{1}{9}$. (2%)
   (b) Use the probability distribution of Y to calculate $P(|Y-\mu|\geq 3\sigma)$. Compare this exact probability with the upper bound provided by Tchebysheff's theorem to see that the bound provided by Tchebysheff's theorem is actually attained when $k=3$. (4%)
   (c) In (b) we guaranteed $E(Y)=0$ by placing all probability mass on the values -1, 0, and 1, with $p(-1)=p(1)$. The variance was controlled by the probabilities assigned to $p(-1)$ and $p(1)$. Using this same basic idea, construct a probability distribution for a random variable X that will yield $P(|X-\mu_X|\geq 2\sigma_X)=1/4$. (4%)

3. Let $m(t)=(1/6)e^{t}+(2/6)e^{2t}+(3/6)e^{3t}$ (moment-generating function). Find the following:
   (a) $E(Y)$. (3%)
   (b) $V(Y)$. (3%)
   (c) the distribution of Y. (4%)

4. Let Z be a standard normal random variable and $W=(Z^2+3Z)^2$.
   (a) Use the moments of Z to derive the mean of W.
   (b) Let $g(Y)$ be a function of the random variable Y, with $E(|g(Y)|)<\infty$. Show that, for every positive constant k,

$$P(|g(Y)|\leq k)\geq 1-\frac{E(|g(Y)|)}{k}.$$

(c) Use the result in (b) to find a value of w that is such that $P(W \le w) \ge 0.90$. (12%)

5. A member of the power family of distributions has a distribution function given by

$$F(y) = \begin{cases} 0, & y < 0 \\ (\frac{y}{\theta})^{\alpha}, & 0 \le y \le \theta \\ 1, & y > \theta \end{cases}$$

where α, $\theta > 0$.

(a) Find the density function.

(b) For fixed values of α and θ, find a transformation $G(U)$ so that $G(U)$ has a distribution function of F when U possesses a uniform (0, 1) distribution.

(c) Given that a random sample of size 5 from a uniform distribution on the interval (0, 1) yielded the values 0.2700, 0.6901, 0.1413, 0.1523, and 0.3609, use the transformation derived in (b) to give values associated with a random variable with a power family distribution with $\alpha = 2$, $\theta = 4$. (12%)

6. Suppose that the length of time Y it takes a worker to complete a certain task has the probability density function given by

$$f(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta \\ 0, & \text{elsewhere} \end{cases}$$

where θ is a positive constant that represents the minimum time until task completion. Let $Y_1, Y_2, \cdots, Y_n$ denote a random sample of completion times from this distribution.

(a) Find the density function for $Y_{(1)} = \min(Y_1, Y_2, \cdots, Y_n)$.

(b) Find $E(Y_{(1)})$. (10%)

7. Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from a population with density function

$$f(y \mid \theta) = \begin{cases} \frac{2\theta^2}{y^3}, & \theta < y < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that $Y_{(1)} = \min(Y_1, Y_2, \cdots, Y_n)$ is sufficient for θ.

(b) Find the MLE for θ.

(c) Find a function of the MLE in part (b) that is a pivotal quantity.

(d) Use the pivotal quantity from part (c) to find a $(1-\alpha)100\%$ confidence interval for θ. (16%)

8. Let $Y_1, Y_2, \cdots, Y_n$ denote a random sample from a Bernoulli-distributed population with parameter p. That is,

$$p(y_i \mid p) = p^{y_i}(1-p)^{1-y_i}, \qquad y_i = 0,\ 1.$$

(a) Suppose that we are interested in testing $H_0 : p = p_0$ versus $H_a : p = p_a$, where $p_0 < p_a$.

i. Show that

$$\frac{L(p_0)}{L(p_a)} = \left(\frac{p_0(1-p_a)}{(1-p_0)p_a}\right)^{\sum y_i} \left(\frac{1-p_0}{1-p_a}\right)^n . \quad (4\%)$$

ii. Argue that $L(p_0)/L(p_a) < k$ if and only if $\sum_{i=1}^n y_i > k^*$ for some constant k^*. (4%)

iii. Give the rejection region for the most powerful test of H_0 versus H_a. (4%)

(b) Is the test derived in (a) uniformly most powerful for testing $H_0 : p = p_0$ versus $H_a : p > p_0$? Why or why not? (6%)