國立高雄大學一百學年度研究所碩士班招生考試試題

科目：統計學
考試時間：100 分鐘

系所：
統計學研究所(風險管理組)
本科原始成績：100 分

是否使用計算機：否

1. (10%) Let the joint density of (X, Y) be $f(x,y) = c$, $x^2 + y^2 \leq 1$.
   (a) Find c and compute the correlation of X and Y.
   (b) Are X and Y independent? Why or why not?

2. (10%) Let X be a random variable with density $f(x) = \lambda e^{-\lambda x}$, $x > 0$, $\lambda > 0$. Find the marginal density functions of $Y = \sqrt{X}$ and $Z = 1 - e^{-\lambda X}$.

3. (10%) A die is rolled and denote the probability of a 6 appearing by p. How many times should we throw the die if the probability of estimating p with error within 0.01 is at least 0.9?

4. (10%) Let $X_1, \cdots, X_n$ be a random sample from the uniform distribution on the interval (-10,10). Let $X_{(1)} \leq \cdots \leq X_{(n)}$ be the order statistics of $X_1, \cdots, X_n$. Compute $\mathrm{E}(X_{(7)} - X_{(3)})$.

5. (15%) Suppose that the random variables $Y_1, \cdots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i,\ i = 1, \cdots, n,$$

   where $x_1, \cdots, x_n$ are fixed constants, and $\varepsilon_1, \cdots, \varepsilon_n$ are i.i.d. $N(0, \sigma^2)$, σ^2 unknown.
   (a) Find a sufficient statistic for (β, σ^2).
   (b) Find the MLE (maximum likelihood estimator) of β. Is it an unbiased estimator?

6. (15%) Let $X_1, \cdots, X_n$ be a random sample from the uniform distribution on the interval $(0, \theta)$, $\theta > 0$. Find the UMVUE (uniformly minimum variance unbiased estimator) of θ if it exists.

7. (30%) Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$ distribution. Consider testing $H_0 : \mu \leq \mu_0$ vs. $H_1 : \mu > \mu_0$. Let $\overline{X}$ denote the sample mean and S^2 denote the sample variance.
   (a) If σ^2 is known, show that the test that rejects H_0 when

$$\overline{X} > \mu_0 + z_\alpha \sqrt{\sigma^2/n}$$

   can be derived as an LRT (likelihood ratio test), where z_α satisfies $P(Z \geq z_\alpha) = \alpha$ with $Z \sim N(0,1)$.
   (b) Show that the test in (a) is a UMP (uniformly most powerful) test.
   (c) If σ^2 is unknown, show that the test that rejects H_0 when

$$\overline{X} > \mu_0 + t_{n-1,\alpha} \sqrt{S^2/n}$$

   can be derived as an LRT, where $t_{n-1,\alpha}$ satisfies $P(T_{n-1} \geq t_{n-1,\alpha}) = \alpha$ with T_{n-1} following the t distribution with $n-1$ degrees of freedom.